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SINGULAR VORTEX*

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During implementation of the PODMODELI-program for the equations of gas dynamics [1] it became clear that the equations have many partial invariant solutions, most of which have not been studied previously. Such solutions require special analysis, which sometimes is not trivial. Our attention was attracted by solutions generated by the rotation group $O(3)$, which is allowed by the equations of gas dynamics. What is specific here is that solutions invariant under $O(3)$ (see, e.g., [2]), which are known as spherically symmetric solutions, are *singular* invariant solutions from the standpoint of group analysis. For the group $O(3)$, however, the necessary conditions for the existence of *nonsingular* partially invariant solutions are rank two and defect one. These solutions are characterized by the fact that their invariant components are spherically symmetric, but their velocity vector component tangential to the sphere is nonzero. It turned out that a fairly broad class of new solutions is opened up here.

The aim of this study is to demonstrate that such solutions do exist and to make a general analysis of them. The kinematics and dynamics of the respective motions of the gas are very involved and the details are not yet very clear. *A singular vortex* is distinguished as an exact solution with a special initial distribution of the tangential component. Particular examples of such exact solutions are given here. In addition, we consider the case of steady-state flow of an incompressible liquid, where solutions of the singular vortex type exist and are fairly foreseeable.

1. Spherical Coordinates. In the space $R^3(\mathbf{x})$, in addition to the Cartesian coordinates of the point $\mathbf{x} = (x, y, z)$ and the corresponding components of the velocity vector $\mathbf{u} = (u, v, w)$, we introduce the spherical coordinates (r, θ, φ) by the formulas

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta \tag{1.1}$$

and the corresponding velocity vector components (U, V, W)

$$\begin{aligned} U &= u\sin\theta\cos\varphi + v\sin\theta\sin\varphi + w\cos\theta, \\ V &= u\cos\theta\cos\varphi + v\cos\theta\sin\varphi - w\sin\theta, \\ W &= -u\sin\varphi + v\cos\varphi. \end{aligned} \tag{1.2}$$

On spheres with $r = \text{const}$ the component U is equal to the normal component of the velocity vector and (V, W) is its tangential component. The vector (V, W) is characterized by its magnitude H and the angle ω of deviation from the meridian:

$$V = H\cos\omega, \quad W = H\sin\omega. \tag{1.3}$$

In these variables the basis operators of the rotation group $O(3)$ allowed by the equations of gas dynamics are

$$\begin{aligned} X_7 &= -\sin\varphi\partial_\theta - \cos\varphi\operatorname{ctg}\theta\partial_\varphi + \cos\varphi(\sin\theta)^{-1}\partial_\omega, \\ X_8 &= \cos\varphi\partial_\theta - \sin\varphi\operatorname{ctg}\theta\partial_\varphi + \sin\varphi(\sin\theta)^{-1}\partial_\omega, \\ X_9 &= \partial_\varphi. \end{aligned} \tag{1.4}$$

Here $\partial_\theta = \partial/\partial\theta$, and so forth.

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In the variables (1.1), (1.2) the initial equations of gas dynamics have the form

$$\begin{gathered} DU + \rho^{-1}p_r = r^{-1}(V^2 + W^2), \\ DV + (\rho r)^{-1}p_\theta = r^{-1}(-UV + \operatorname{ctg}\theta W^2), \\ DW + (\rho r\sin\theta)^{-1}p_\varphi = r^{-1}(-UW - \operatorname{ctg}\theta VW), \\ D\rho + \rho \operatorname{div} \mathbf{u} = 0, \quad DS = 0, \quad p = f(\rho, S), \end{gathered} \tag{1.5}$$

where ρ is the density, p is the pressure, S is the entropy of the gas, and D and div are operator

$$\begin{gathered} D = \partial_t + U\partial_r + r^{-1}V\partial_\theta + (r\sin\theta)^{-1}W\partial_\varphi, \\ \operatorname{div}\mathbf{u} = r^{-2}(r^2U)_r + (r\sin\theta)^{-1}(\cos\theta V + \sin\theta V_\theta + W_\varphi) \end{gathered} \tag{1.6}$$

(the indices denote the respective partial derivatives). The function f in (1.5) specifies the equation of state for the gas.

2. Representation of Partially Invariant Solutions. In the space of the variables t, r, θ, φ, U, H, ω, ρ, S the group $O(3)$ with operators (1.4) has the invariants t, r, θ, U, H, ρ, S. The desired quantity ω here is a "root" of the function. According to [3], therefore, partially invariant solutions of rank two and defect one are possible, in which U, H, ρ, S depend only on t and r, and ω generally is a function of all the independent variables t, r, θ, φ. The initial representation of the desired partially invariant solutions thus has the form

$$U = U(t,r),\ H = H(t,r),\ \rho = \rho(t,r),\ S = S(t,r),\ \omega = \omega(t,r,\theta,\varphi). \tag{2.1}$$

When this representation is substituted into the system (1.5) and the second and third equations are combined the system breaks up into two subsystems: an invariant subsystem

$$D_0U + \rho^{-1}p_r = r^{-1}H^2, \quad D_0(rH) = 0,\ D_0S = 0 \qquad (p = f(\rho, S)) \tag{2.2}$$

with the operator $D_0 = \partial_t + U\partial_r$, and a supplementary subsystem

$$\begin{aligned} k\sin\theta D_0\omega + \sin\theta\cos\omega\,\omega_\theta + \sin\omega\,\omega_\varphi &= -\cos\theta\sin\omega, \\ \sin\theta\sin\omega\,\omega_\theta - \cos\omega\,\omega_\varphi &= \cos\theta\cos\omega + h\sin\theta, \end{aligned} \tag{2.3}$$

where we have introduced auxiliary functions, which depend only on t and r:

$$k = r/H, \quad h = k(\rho^{-1}D_0\rho + r^{-2}(r^2U)_r). \tag{2.4}$$

Here and below we assume that $H \neq 0$. If $H = 0$, then by (1.3) the tangential component of the velocity vector is zero and the system (1.5) is transformed into the familiar system of equations of the spherically symmetric motions of a gas. New results exist, therefore, only if the redefined system (2.3) of two equations for one "root" of the function ω has solutions.

To prove that the system (2.3) has solutions it is convenient to look for them in the implicit form

$$\Phi(t,r,\theta,\varphi,\omega) = 0 \qquad (\Phi_\omega \neq 0). \tag{2.5}$$

Then the function Φ should be an invariant of two linear differential operators obtained from Eqs. (2.3):

$$\begin{aligned} \Omega_1 &= k\sin\theta D_0 + \sin\theta\cos\omega\,\partial_\theta + \sin\omega\,\partial_\varphi - \cos\theta\sin\omega\,\partial_\omega, \\ \Omega_2 &= \sin\theta\sin\omega\partial_\theta - \cos\omega\partial_\varphi + (\cos\theta\cos\omega + h\sin\theta)\partial_\omega. \end{aligned} \tag{2.6}$$

Calculation of the commutator of these operators leads to the identity

$$[\Omega_1, \Omega_2] + \cos\theta \sin\omega\Omega_1 - (\cos\theta\cos\omega + h\sin\theta)\Omega_2 = \sin^2\theta(kD_0h - h^2 - 1)\partial_\omega.$$

It follows that the operators (2.6) are in involution if and only if the functions k and h satisfy the equation

$$kD_0h = h^2 + 1. \tag{2.7}$$

This equation along with (2.4) complements the invariant subsystem (2.2) to a system that is closed in the unknowns U, H, ρ, S, and h. By (2.7) the system (2.3) is passive and its general solution depends on an arbitrary function of two variables.

To construct the general solution of the system (2.3) we replace t and r by two independent variables, the Lagrangian coordinate $\xi = \xi(t, r)$ and modified time $\tau = \tau(t, r)$, in accordance with the equations

$$D_0\xi = 0, \quad \xi(0,r) = r, \quad kD_0\tau = 1, \quad \tau(0,r) = 0. \tag{2.8}$$

Then $kD_0 = \partial_\tau$. In this case Eq. (2.7) is integrated and with the condition $h(0, r) = 0$ it gives $h = \tan\tau$. When we go over from the operators (2.6) to the operators Ω_3 and Ω_4 from the formulas $\sin\theta\Omega_3 = \cos\omega\Omega_2$ and $\Omega_4 = \sin\omega\Omega_1 - \cos\omega\Omega_2$ and replace ω by

$$\eta = \cos\tau\sin\theta\cos\omega - \sin\tau\cos\theta \tag{2.9}$$

the new operators are

$$\Omega_3 = \cos\omega\partial_\tau + \partial_\theta, \quad \Omega_4 = \sin\theta\sin\omega\partial_\tau + \partial_\varphi. \tag{2.10}$$

Two invariants of the system of operators (2.6) have already been found: they are ξ and η. The system (2.10) is integrated to find the third invariant ζ, which defined implicitly by

$$\sqrt{1-\eta^2}\sin(\zeta + \varphi) = \cos\tau\cos\theta\cos\omega + \sin\tau\sin\theta. \tag{2.11}$$

Accordingly, the general solution of the system (2.3), where $kD_0 = \partial_\tau$ and $h = \tan\tau$, expressed in the implicit form (2.5), is

$$F(\xi, \eta, \zeta) = 0 \tag{2.12}$$

with an arbitrary function F.

3. Analysis of the Solution. The problem of constructing the solution of the system (2.3) can be considered from a different standpoint. The first of these equations is evolutionary relative to the modified time τ:

$$\omega_\tau = -\cos\omega\,\omega_\theta - (\sin\omega/\sin\theta)\omega_\varphi - \operatorname{ctg}\theta\sin\omega. \tag{3.1}$$

Its solution $\omega(\tau, \theta, \varphi)$ is determined by setting the initial condition

$$\omega(0,\theta,\varphi) = \omega_0(\theta,\varphi). \tag{3.2}$$

If with this solution we set

$$N(\tau,\theta,\varphi) = \sin\theta\sin\omega\omega_\theta - \cos\omega\omega_\varphi - \cos\theta\cos\omega - \operatorname{tg}\tau\sin\theta,$$

the second equation in (2.3) is $N = 0$. It does not contain the derivative with respect to τ and, therefore, imposes a *necessary* condition on the function (3.2), namely,

$$N(0,\theta,\varphi) = 0. \tag{3.3}$$

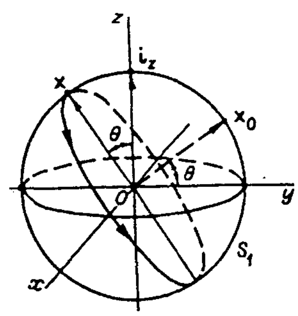

Fig. 1

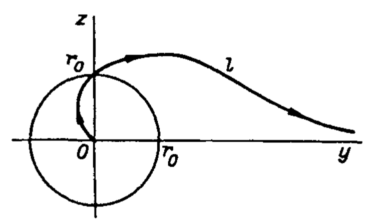

Fig. 2

Assumption 1. *The solutions of Eq.* (3.1) *satisfy the identity*

$$\sin\theta N_\tau + \cos\theta\cos\omega N_\theta + \sin\omega N_\varphi = N^2 + \operatorname{ctg}\theta\cos\omega N.$$

By virtue of this identity it follows from (3.3) that $N(\tau,\ \theta,\ \varphi)$ for all τ, i.e., the condition (3.3) is also *sufficient*.

Solving Eq. (3.1) by the method of characteristics reduced to integrating the system of ordinary differential equations

$$\frac{d\theta'}{d\tau} = \cos\omega', \quad \frac{d\varphi'}{d\tau} = \frac{\sin\omega'}{\sin\theta'}, \quad \frac{d\omega'}{d\tau} = -\operatorname{ctg}\theta'\sin\omega' \tag{3.4}$$

with the initial data for $\tau = 0$

$$\theta'(0) = \theta, \quad \varphi'(0) = \varphi, \quad \omega'(0) = \omega_0(\theta,\varphi), \tag{3.5}$$

where the function ω_0 is the same as in (3.2). By Assumption 1 this solution gives rise to the solution of the entire system (2.3), if and only if ω_0 satisfies Eq. (3.3).

On the other hand, the equations of the particle trajectories $dx = dt = \mathbf{u}$, written in spherical variables (1.1)-(1.3) with modified time τ, have the form

$$\frac{dr}{d\tau} = kU, \quad \frac{d\theta}{d\tau} = \cos\omega, \quad \frac{d\varphi}{d\tau} = \frac{\sin\omega}{\sin\theta}. \tag{3.6}$$

Here $\omega = \omega(\tau,\ \theta,\ \varphi)$; the last two equations coincide with the first two equations of the system (3.4). Hence the characteristics of Eq. (3.1) are radial projections of the gas particle trajectories onto a unit sphere S_1 ($r = 1$). The solution of the system (3.4), therefore, describes *spherical trajectories of particles* and the evolution of the angle ω along those trajectories.

It is confirmed directly that the invariants η and ζ from (2.9) and (2.11) as well as the quantity $\sigma = \sin\theta'\sin\omega'$ retain constant values along the characteristics. The equation of the characteristics (3.4), which determine spherical particle trajectories, are integrated automatically in the form

$$\cos\theta' = \cos\tau\cos\theta - \sin\tau\sin\theta\cos\omega,$$
$$\sin\theta'\sin(\varphi' - \varphi) = \sin\tau\sin\omega, \quad \sin\theta'\sin\omega' = \sin\theta\sin\omega, \tag{3.7}$$

where $\omega = \omega_0(\theta, \varphi)$.

We obtain a clear geometric representation of the behavior of spherical particle trajectories, if we turn to Cartesian coordinates on a sphere S_1, setting $r = 1$ in Eqs. (1.1). Then the point (θ, φ) is specified by the vector $\mathbf{x} = (x, y, z)$ and the point (θ', φ'), by the vector $\mathbf{x}' = (x', y', z')$. In this notation Eqs. (3.7) lead to the following description of motion on the sphere S_1:

$$R\mathbf{x}'(\tau) = l(\tau)\mathbf{x} + \mathbf{m}\sin\tau \qquad (R = \sqrt{x^2 + y^2}). \tag{3.8}$$

Here

$$l = \cos\tau\sin\theta + \sin\tau\cos\theta\cos\omega, \ \mathbf{m} = (-y\sin\omega, x\sin\omega, -\cos\omega). \tag{3.9}$$

Assumption 2. *During motion* (3.8) *the spherical trajectory of any point* $\mathbf{x}$ *is a great circle* (*geodesic circle*) *on* S_1 *and its translation* (*relative to* τ) *is equal to unity.*

As proof it is sufficient to note that there is only one (apart from sign) vector $\mathbf{x}_0$ that is independent of the time τ and satisfies the equation (the symbol $\mathbf{a}\cdot\mathbf{b}$ denotes the scalar product of vectors $\mathbf{a}$ and $\mathbf{b}$)

$$\mathbf{x}' \cdot \mathbf{x}_0 = 0 \qquad (|\mathbf{x}_0| = 1). \tag{3.10}$$

Indeed, for any τ by virtue of (3.8) Eq. (3.10) reduces to the system

$$\mathbf{x} \cdot \mathbf{x}_0 = 0, \quad \mathbf{m} \cdot \mathbf{x}_0 = 0 \qquad (|\mathbf{x}_0| = 1),$$

the only solution of which is given by

$$R\mathbf{x}_0 = (-y, x, 0)\cos\omega + (-xz, -yz, R^2)\sin\omega. \tag{3.11}$$

The vector $\mathbf{x}_0$ indicates the point on S_1, called the *pole* of the spherical trajectory (great circle), whose plane is perpendicular to $\mathbf{x}_0$.

4. Initial Distribution. By virtue of the above the final solution of the system (2.3) comes down to finding the function $\omega_0(\theta, \varphi)$, which should be solved by Eq. (3.3). Direct substitution confirms that this equation has two independent integrals, which are obtained from (2.9) and (2.11) for $\tau = 0$:

$$\eta_0 = \sin\theta\cos\omega_0, \quad \sqrt{1 - \eta_0^2}\sin(\zeta_0 + \varphi) = \cos\theta\cos\omega_0. \tag{4.1}$$

The sought solution ω_0 can be found in implicit form from a complementary relation of the form $\eta_0 = \chi(\zeta_0)$ with an arbitrary given function χ. Therefore, ω_0 is determined to within an arbitrary function of one argument.

The function $\omega_0(\theta, \varphi)$ should be single-valued wherever it is defined (apart from the poles $\theta = 0$ and $\theta = \pi$). Isolating specific solutions may also require that the function be definite on the entire sphere S_1 (*completeness* property).

Assumption 3. *The completeness property for* $\omega_0 = \pi/2$ *on the entire sphere* S_1, *apart from the poles.*

Indeed, the first equation of (4.1) shows that it should be necessary that $\eta_0 = 0$. Then $\cos\omega_0 = 0$ and $\omega_0 = \pi/2$. The second equation of Eq. (4.1) is satisfactory for $\zeta_0 = -\varphi$. This means that $\chi \equiv 0$ in the complementary relation $\eta_0 = \chi(\zeta_0)$.

5. Singular Vortex. The initial value $\omega_0 = \pi/2$ determines a specific motion, which we shall call a *singular vortex*. The substitution $\omega_0 \equiv \pi/2$ into the equation of spherical trajectories (3.8) leads to

$$\mathbf{x}' = \mathbf{x}\cos\tau + R^{-1}(\mathbf{i}_z \times \mathbf{x})\sin\tau \qquad (R = \sqrt{x^2 + y^2}), \tag{5.1}$$

where the unit vector $\mathbf{i}_z$ indicates the southern poles ($z = 1$) and $\times$ is the symbol of the vector product. The pole of the trajectory (5.1) is found from (3.11) for $\omega = \pi/2$:

$$\mathbf{x}_0 = R^{-1}(-xz, -yz, R^2). \tag{5.2}$$

The solution obtained is invariant under rotations about the z axis. The vectors $\mathbf{x}$, $\mathbf{x}_0$, and $\mathbf{i}_z$ lie in one diametral plane, and $\mathbf{x}\cdot\mathbf{x}_0 = 0$. The relative position of the spherical trajectory that passes through a typical point $\mathbf{x}$ and its pole $\mathbf{x}_0$ is shown in Fig. 1.

The representation of the spherical trajectories of particles reveals some distinctive features of the motion of a gas in a singular vortex. For example, by (5.1) we have $z' = z\cos\tau$. Thereby, for $\tau > 0$ the points $\mathbf{x}'$ fill the spherical band $\tau < \theta' < \pi - \tau$ and not the entire sphere S_1. Moreover, calculation of the component $\Omega(\mathbf{x}_0) = \mathbf{x}_0\cdot\mathrm{rot}\,\mathbf{u}(\mathbf{x}_0)$ that is normal to S_1 shows that the initial distribution of the velocity field at any point $\mathbf{x}_0$, other than the poles, is $\Omega(\mathbf{x}_0) = 0$, i.e., this is a surface vortex-free distribution. For the poles $\mathbf{x}_0 = (0, 0, \pm 1)$, however, $\Omega(\mathbf{x}_0) = \infty$.

6. Radial Motion. In the analysis above the value of the Lagrangian coordinate ξ (invariant of the system (2.3)) remained an arbitrary fixed value. The complete picture of the motion, in particular the physical trajectories of the gas particles, can be represented only with allowance for the radial motion, which is described by the solution of the invariant subsystem (2.2), (2.4), (2.8). The second and third equations of (2.2) are integrated in the form $rH = H_0(\xi)$, $S = S_0(\xi)$ with arbitrary functions H_0 and S_0 of the Lagrangian coordinate ξ. This leaves the following subsystem of equations of motion for the desired functions U, ρ, ξ, τ of the variables t, r:

$$\begin{gathered} U_t + UU_r + \rho^{-1}p_r = r^{-3}H_0^2(\xi), \\ \rho^{-1}(\rho_t + U\rho_r) + r^{-2}(r^2U)_r = r^{-2}H_0(\xi)\mathrm{tg}\,\tau, \\ \xi_t + U\xi_r = 0, \qquad \tau_t + U\tau_r = r^{-2}H_0(\xi). \end{gathered} \tag{6.1}$$

Here $p = f(\rho, S_0(\xi))$ with a given function f. The initial data for the system (6.1) have the form

$$U(0,r) = U_0(r), \quad \rho(0,r) = \rho_0(r), \quad \xi(0,r) = r, \quad \tau(0,r) = 0. \tag{6.2}$$

In the system (6.1) the functions $H_0(\xi) \neq 0$ and $S_0(\xi)$ are considered to be preset arbitrarily. Presetting these functions distinguishes a specific class of gas motions.

For example, with the choice $H_0 = L\xi^n$, where L, n = const and S_0 = const, the system (6.1) has self-similar solutions, in which the desired U, ρ, p, τ depend only on $\lambda = r/t$, a $\xi = t^{1/n}\sigma(\lambda)$. In those variables (6.1) reduces to a system of ordinary differential equations

$$\begin{gathered} (U-\lambda)U' + \rho^{-1}p' = L^2\lambda^{-3}\sigma^{2n}, \\ \lambda^2\rho^{-1}(U-\lambda)\rho' + (\lambda^2U)' = L\sigma^n\mathrm{tg}\,\tau, \\ n(U-\lambda)\sigma' + \sigma = 0, \qquad \lambda^2(U-\lambda)\tau' = L\sigma^n, \end{gathered}$$

where $p = f(\rho, S_0)$ and the prime denotes the derivative with respect to λ.

For an special equation of state of the form $p = A\rho + B$ with constant A and B there exists a solution of the system (6.1), which describes isentropic motion (S_0 = const), in which $U = 0$, i.e., the gas particles have no radial motion. This solution is given by

$$\xi = r, \quad H_0 = \mu r^2, \quad \tau = \mu t, \quad \rho = \rho_0(r)/\cos\mu t. \tag{6.3}$$

Here

$$\rho_0(r) = \rho_{00}\exp((\mu^2/2A)(r^2 - r_0^2))$$

with the constants $\mu > 0$, ρ_{00}, r_0. If as the initial configuration we take a spherical layer $0 < r_1 < r < r_2$ and assign a distribution of the singular vortex type, i.e., $\omega = \pi/2$ for any r from the interval (r_1, r_2), then with time this spherical layer is compressed in the form of a torus-like body and collapses to a ring with given radii $r_1 < r_2$ in the plane $z = 0$ at the time $t = \pi/2\mu$.

It is useful to note that when $H_0 = \text{const} \neq 0$ extension of the variables leads to $H_0 = \pm 1$.

A list of other cases of simplification of the system (6.1), giving new forms of a special vortex, can be made on the basis of a complete group analysis of the system.

7. Steady-State Flow in an Incompressible Liquid. All of the conclusions in Sections 1-5 are valid in this case with the following additions. There is no Lagrangian coordinate ξ since the solution is independent of the time t. Therefore, $H_0 =$ const and we can choose $H_0 = 1$. Without loss of generality, we assume that $\rho = 1$. The particle trajectories coincide with the field lines, along which r can be taken to be a parameter. The representation (5.1) of spherical trajectories (field lines) on the sphere S_1 in a singular vortex remains valid. The differential equations of radial motion (6.1) reduce to

$$UU_r + p_r = \frac{1}{r^3}, \quad (r^2U)_r = \operatorname{tg}\tau, \quad r^2U\tau_r = 1. \tag{7.1}$$

The first of these equations gives the Bernoulli integral

$$\frac{1}{2}U^2 + \frac{1}{2r^2} + p = b \qquad (b = \text{const}), \tag{7.2}$$

which defines the pressure p. For the remaining system of the last two equations of (7.1) we introduce the initial data r_0, U_0, with which $\tau(r_0) = 0$, $U(r_0) = U_0 > 0$.

It can be easily checked that the system (7.1) admits the extension operator $r\partial_r - U\partial_U - 2p\partial_p$. Without loss of generality, therefore, we can assume that $r_0^2U_0 = 1$. With this proviso the solution of the system (7.1) is

$$\sin\tau = \operatorname{th}(r - r_0), \quad r^2U = \operatorname{ch}(r - r_0). \tag{7.3}$$

The singular vortex in this solution is distinguished by the value $\omega_0 = \pi/2$ being given on the sphere S_0 ($r = r_0$), which corresponds to $\tau = 0$.

A parametric representation of the field lines is obtained from (5.1) with allowance for (7.3). The field passing through the point $(x_0, y_0, z_0) \in S_0$, which is described by

$$\begin{gathered} x = \frac{x_0}{r_0}\frac{r}{\operatorname{ch}(r - r_0)} - \frac{y_0}{R_0}r\operatorname{th}(r - r_0), \\ y = \frac{y_0}{r_0}\frac{r}{\operatorname{ch}(r - r_0)} + \frac{x_0}{R_0}r\operatorname{th}(r - r_0), \quad z = \frac{z_0}{r_0}\frac{r}{\operatorname{ch}(r - r_0)}, \end{gathered} \tag{7.4}$$

where $r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$; $R_0 = \sqrt{x_0^2 + y_0^2}$.

Each field line (7.4) lies in the plane $x_0x + y_0y = (R_0^2/z_0)z$. A typical field line l, lying in the plane $x = 0$, is shown in Fig. 2. All of the other field lines are obtained by rotations, first about the y axis and then about the z axis. This gives a general idea of the region occupied by liquid in a flow of the singular vortex type. At the center ($r = 0$) is a conical vortex-source combination with a 2π flow volume. The liquid flows through the sphere S_0 and spreads in the form of a layer, which contracts to the plane $z = 0$. A torus-like plane, in which there is no liquid, exists near the segment $|z_0| < r_0$.

In summary, a preliminary analysis has shown that the motion in a singular vortex is very complex. In the case of motion of a gas new peculiar features appear because in the given class of solutions the physical fields of velocity, pressure, and density cannot be extended beyond possible barriers (e.g., into the region of negative pressures). Many questions about the details of the motion of a gas in a singular vortex, therefore, remain open.

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